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AGGREGATION OF UTILITY FUNCTIONS

Edmund Eisenberg

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14 July 1958

Approved for OTS release

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### SUMMARY

It is shown that if in an economy each consumer has a fixed income and acts so as to maximize a concave, continuous and homogeneous utility function, then both a social welfare and a community utility function exist.

## AGGREGATION OF UTILITY FUNCTIONS

1. Introduction. The concepts of economic equilibrium and aggregation of utilities have been discussed separately in various papers. While equilibrium has been shown to exist under very general assumptions (see [1]), the community revealed preference may, under these same assumptions, possess intransitivities; in which case one knows a priori that a community (or aggregate) preference ordering does not exist.

An apparently related question is that of a community pseudo-utility (or social welfare function). There we ask whether there is an explicit function of the individual's utilities (and independent of prices) which is maximized at equilibrium.

The purpose of this paper is to show that if every member of a community has a fixed income and acts according to a concave, continuous and homogeneous utility function then both a community utility and a social welfare function exist. Thus one is able to define, unambiguously, the index of the community standard of living as well as the price index.

For the reader who is interested in practical calculations of equilibrium distributions and prices, these are shown to be, in our model, the primal and dual variables of a concave programming problem with linear constraints. With the objective function given explicitly in terms of the individual's utilities one can use any of the known methods of concave programming to

calculate the desired quantities.

Our basic tool is the saddle point theorem for concave programming (see for example Theorem 1 of [4]). We do not, however, require the functions in question to be differentiable.

2. Definitions and Assumptions. In the model to be discussed we have  $m$  buyers  $B_1, \dots, B_m$  and  $n$  goods  $G_1, \dots, G_n$ . A bundle of goods is a vector  $x = (\xi_1, \dots, \xi_n)$  in real  $n$ -space with  $\xi_j \geq 0$  for each  $j$ ;  $\xi_j$  represents a quantity of  $G_j$ . We further assume that each  $B_i$  has a fixed positive income  $\beta_i$ , and that we have chosen our monetary unit in such a manner that  $\sum_{i=1}^m \beta_i = 1$ . Finally, each buyer,  $B_i$ , has a real valued non-constant utility function  $u_i(x)$  defined for all bundles  $x$ , with each  $u_i$  being concave, continuous, homogeneous of order 1<sup>1</sup> and  $u_i(x) \geq 0$  for all bundles  $x$ .

Before proceeding further an explanation of the above is called for. The requirement of fixed income, is of course not as general as one would wish; it has however wide applications and it is this assumption, together with homogeneity, which enables us to accomplish the tasks outlined in the introduction.

The fact that if in the above model we remove the requirement that each  $u_i$  be homogeneous then the community demand need not be rational is well known. One need not go very far to construct examples with an intransitive community preference.

For a given set of non-negative prices  $p = (\pi_1, \dots, \pi_n)$

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<sup>1</sup>See Section 3.

the demand set of  $B_1$ , denoted  $D_1(p)$ , is the set of all bundles which maximize  $B_1$ 's utility without exceeding his personal budget. More formally:

$$D_1(p) = \{x | u_1(x) \text{ is maximal subject to: } x \geq 0 \text{ and } xp \leq \beta_1^2\}.$$

In general,  $D_1(p)$  may be the empty set; however when prices are all positive it follows from the continuity of  $u_1$  that  $D_1(p)$  is non-empty. We also define the community demand to be

$$D^*(p) = \left\{x \mid x = \sum_{i=1}^m x_i, x_i \in D_i(p) \text{ for all } i = 1, \dots, m\right\}.$$

In words,  $D^*(p)$  is the set of all those bundles that are demanded at prices  $p$ , by the community as a whole.

The community-pseudo-utility function  $\Psi$ , denoted C.P.U., is defined by

$$\Psi(x_1, \dots, x_m) = \prod_{i=1}^m [u_i(x_i)]^{\beta_i}$$

for every collection of  $m$  bundles  $x_1, \dots, x_m$ .

The community utility function  $u$ , denoted C.U., is defined by

$xp$  is the inner product of the vectors  $x$  and  $p$ , while  $x \geq 0$  means that every component of  $x$  is non-negative.

$$u(x) = \text{Sup} \left\{ \psi(x_1, \dots, x_m) \mid x_1, \dots, x_m \text{ are bundles and } \sum_{i=1}^m x_i \leq x^3 \right\}.$$

Our goal is to demonstrate, among other results, that  $u$  is a true aggregate utility function and that  $\psi$  is a social welfare function for the model under consideration. The latter result is given in Theorem 3\* while the former means that for every set of non-negative prices  $p, D^*(p)$  is precisely the set demanded with utility  $u$  and income 1. Formally, we must show that if  $p = (p_1, \dots, p_u) \geq 0$  then

$$D^*(p) = \{x \mid u(x) \text{ is maximal subject to } x \geq 0 \text{ and } xp \leq 1\}.$$

The principal results of this paper are:

**Theorem 1.** The function  $u$  is a true aggregate utility function for the model.

**Theorem 2.** If  $x \in D^*(p)$  and  $u(x) = \psi(x_1, \dots, x_m)$  then  $x_i \in D_i(p)$  for  $i = 1, \dots, m$ . Conversely, if  $x_i \in D_i(p)$  for  $i = 1, \dots, m$  and  $x = \sum_{i=1}^m x_i$  then  $u(x) = \psi(x_1, \dots, x_m)$ .

**Theorem 3.** If  $x$  is a bundle with each component positive and  $u(x) = \psi(x_1, \dots, x_m)$  then there exist price  $p$  such that  $x \in D^*(p)$ .

Theorem 1 is self explanatory, Theorems 2 and 3 may be

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\*See Section 3.

interpreted in terms of equilibrium properties as follows:

Given a bundle  $x$  then the collection of bundles  $x_1, \dots, x_m$  and the price vector  $p$  are called an equilibrium distribution and prices for  $x$ , respectively, providing  $p$  is a non-negative vector,  $x_i \in D_i(p)$  all  $i = 1, \dots, m$ ,  $\sum_{i=1}^m x_i \leq x$ , and  $px \leq 1$ . This is the standard definition of equilibrium when  $x$  represents quantities of the goods  $G_1, \dots, G_n$  available in the economy, i.e. at prices  $p$  every buyer has maximized his utility, there is sufficient supply of each goods in the economy to meet the demand and only free goods (those with zero price) can be in oversupply. Also, for a fixed bundle  $x$ , a collection of bundles  $x_1, \dots, x_m$  is called a maximizing distribution for  $x$  providing  $u(x) = V(x_1, \dots, x_m)$ .

Theorems 2 and 3 are then equivalent to (as is seen trivially by using Theorem 1 and comparing the two expressions for  $D^*(p)$ )

Theorem 2\*. If for the bundle  $x$  there exists an equilibrium distribution and prices then every maximizing distribution is an equilibrium distribution. Conversely, if  $x_1, \dots, x_m$  is an equilibrium distribution for the bundle  $x$  then it is a maximizing distribution.

Theorem 3\*. If  $x$  is a bundle with each component positive then every maximizing distribution is an equilibrium distribution.

At first glance one may wonder why Theorem 2\* does not say that every maximizing distribution is an equilibrium distribution.

This is simply not true. For, as is the case when  $x$  is identically zero, the set of equilibrium distributions is empty while the set of maximizing distributions, in view of the continuity of the C.P.U., is never empty. Theorem 3\* does, of course, guarantee that an equilibrium distribution exists in case  $x$  is positive. It should be mentioned that Theorem 3\* has been proved, under much weaker assumptions elsewhere (see [1]). One of the reasons for its inclusion here is that other methods of proof depend on general fixed-point theorems while here use is made, essentially, of separation theorems for convex sets.

Finally, certain uniqueness questions can be answered rather easily in the framework of our formulation. Equilibrium prices need not be unique without further assumptions, and certainly one cannot expect equilibrium distributions to be unique. However, the satisfaction of each buyer at equilibrium, or the pay-off, is unique.

Theorem 4. Suppose that for a given bundle  $x$  there are two equilibrium price vectors  $p$  and  $q$  with  $x_1, \dots, x_m$  and  $y_1, \dots, y_m$  being the corresponding equilibrium distributions. Then  $u_i(x_i) = u_i(y_i)$  for every  $i=1, \dots, m$ .

3. Proof of Theorems. We shall first prove several lemmas. All matrices and vectors discussed have real components and we use the conventional notation for matrix multiplication. If  $A = (a_{ij})$  is an  $m \times n$  matrix,  $x = (\xi_1, \dots, \xi_m)$

$y = (\eta_1, \dots, \eta_n)$  are vectors, then  $xA$  stands for the vector  $(\sum_{i=1}^m \xi_i \alpha_{i1}, \dots, \sum_{i=1}^m \xi_i \alpha_{in})$  and  $Ay$  stands for the vector  $(\sum_{j=1}^n \eta_j \alpha_{1j}, \dots, \sum_{j=1}^n \eta_j \alpha_{mj})$  while  $xAy$  stands for the inner product of  $xA$  and  $y$  (or equivalently  $x$  and  $Ay$ ) i.e.  
 $xAy = \sum_{j=1}^n \sum_{i=1}^m \xi_i \eta_j \alpha_{ij}$ . A vector inequality means that the same inequality obtains componentwise. By  $R_+^k$  is meant the set of real  $k$ -tuples  $x = (\xi_1, \dots, \xi_k)$  with  $\xi_i \geq 0$   $i=1, \dots, k$ . A function  $\phi : R_+^k \rightarrow R$  ( $R$  being the set of real numbers) is concave providing  $\phi(\lambda x + (1-\lambda)y) \geq \lambda \phi(x) + (1-\lambda)\phi(y)$  for all  $x, y \in R_+^k$  and all  $\lambda$  in the real interval  $[0, 1]$ ;  $\phi$  is positively homogeneous of order  $r$  (or, for short,  $r$ -homogeneous) providing  $r$  is a real number and  $\phi(\lambda x) = \lambda^r \phi(x)$  for all  $x \in R_+^k$  and  $\lambda > 0$ ; when saying  $\phi$  is continuous we shall always mean with respect to the relative topology of  $R_+^k$  as a subspace of  $R^k$ . We also say that  $\phi$  is quasi-concave providing  $\phi(\lambda x + (1-\lambda)y) \geq \alpha$  whenever  $\phi(x), \phi(y) \geq \alpha$  and  $\lambda \in [0, 1]$ .

Lemma 1.

Let  $A$  be an  $m \times n$  matrix,  $b \in R^n$ ,  $\phi : R_+^m \rightarrow R$  a concave function. Suppose  $A, b$  have the property that for some  $x \in R_+^m$  we have  $xA < b$ , and suppose  $x_0$  has the property that  $\phi(x_0)$  is maximal subject to  $x_0 \geq 0, x_0 A \leq b$ .

Conclusion: There is a vector  $y_0 \in R_+^n$  such that

- (1)  $x_0 A y_0 = b y_0$  and
- (2)  $\phi(x_0) \geq \phi(x) + (x_0 - x) A y_0$  for all  $x \in R_+^m$ .

Proof: By the maximality of  $\phi(x_0)$  the inequalities:

$$x \in R_+^m$$

$$xA \leq b$$

$$\phi(x_0) - \phi(x) < 0$$

have no solution. Whence it follows (see Theorem 1 of [2]) that there is a  $y \in R_+^n$  and a real number  $\eta \geq 0$  such that  $y$  and  $\eta$  are not both zero and

$$xAy - by + \eta [\phi(x_0) - \phi(x)] \geq 0 \text{ for all } x \in R_+^m.$$

If  $\eta = 0$  then  $y \neq 0$  and  $xAy \geq 0$  for all  $x \geq 0$ , but we assumed that  $xA < 0$  for some  $x \geq 0$  hence  $xAy < 0$  unless  $y = 0$ , both of which are impossible. Thus  $\eta \geq 0$ , let  $y_0 = \frac{1}{\eta}y$  then  $y_0 \in R_+^n$  and

$$xAy_0 - by_0 + \phi(x_0) - \phi(x) \geq 0, \quad \text{for all } x \in R_+^m$$

or

$$\phi(x_0) \geq \phi(x) + (b - xA)y_0, \quad \text{for all } x \in R_+^m$$

Letting  $x = x_0$  in the above we get

$$x_0 A y_0 \geq b y_0$$

but  $x_0 A \leq b$  and  $y_0 \in R_+^m$ , hence  $x_0 A y_0 \leq b y_0$ . Thus  $x_0 A y_0 = b y_0$ , completing the proof.

Lemma 2.

Suppose  $\alpha, \beta, \gamma$  are real numbers with the property that

$$(3) \quad \alpha(1-\lambda^\beta) \geq \gamma(1-\lambda), \text{ for all } \lambda \text{ in some open neighborhood of } 1.$$

Then  $\alpha\beta = \gamma$ .

Proof: For  $0 \leq \lambda < 1$  we have

$$\alpha \left( \frac{1-\lambda^\beta}{1-\lambda} \right) \geq \gamma$$

while for  $1 < \lambda$  we have

$$\alpha \left( \frac{1-\lambda^\beta}{1-\lambda} \right) \leq \gamma$$

But,  $\lim_{\lambda \rightarrow 1} \frac{1-\lambda^\beta}{1-\lambda} = \beta$  (as may be seen, for instance, by differentiating the numerator and denominator of the quotient  $\frac{1-\lambda^\beta}{1-\lambda}$ )

with respect to  $\lambda$ ). Thus  $\gamma \leq \alpha\beta \leq \gamma$  and  $\alpha\beta = \gamma$ .

Lemma 3.

If  $A, b, \phi, x_0$  satisfy the assumptions of Lemma 1 and in addition  $\phi$  is 1-homogeneous then  $\phi(x_0) = x_0 A y_0$  and, thus, (2) may be written

$$(3) \quad x A y_0 \geq \phi(x) \quad \text{for all } x \in R_+^m.$$

Proof: Let  $\lambda$  be a non-negative real number. In (2) let  $x = \lambda x_0$ , we then have

$$\begin{aligned} \phi(x_0) &\geq \phi(\lambda x_0) + (x_0 - \lambda x_0) A y_0 \\ &= \lambda \phi(x_0) + (1-\lambda) x_0 A y_0 \end{aligned}$$

$$\text{or} \quad (1-\lambda) \phi(x_0) \geq (1-\lambda) x_0 A y_0$$

Applying Lemma 2 we have  $\phi(x_0) = x_0 A y_0$ .

Lemma 4.

Let  $\beta_1, \dots, \beta_m$  be positive numbers satisfying  $\sum_{i=1}^m \beta_i = 1$ . Then for any  $a = (\alpha_1, \dots, \alpha_m) \in R_+^m$  we have

$$(4) \quad \sum_{i=1}^m \alpha_i^{\beta_i} \leq \alpha \sum_{i=1}^m \beta_i^{\beta_i} \quad (\text{where } \alpha = \sum_{i=1}^m \alpha_i)$$

and equality holds in (4) if and only if  $\alpha_1 = \alpha \beta_1$ .

Proof: If  $\alpha = 0$  (i.e.  $a = 0$ ) then the conclusion is trivial. Suppose  $\alpha > 0$ , let  $\gamma_1 = \frac{\alpha_1}{\alpha}$  and consider the function  $f(c) = \prod_{i=1}^m \gamma_i^{\beta_i}$  defined for all  $c = (\gamma_1, \dots, \gamma_m) \in \mathbb{R}_+^m$  for which  $\sum \gamma_i = 1$ . This function is continuous on a compact set hence achieves its maximum at  $c_0 = (\gamma_1^0, \dots, \gamma_m^0)$ . Note that  $c_0 > 0$  for otherwise  $f(c_0) = 0$  which clearly is not the maximum value of  $f$ . But  $f$  is differentiable, hence all partial derivatives of  $f$  are zero at  $c_0$ , as are all partials of

$$\log [f(c)] = \sum_{i=1}^m \beta_i \log \gamma_i = \sum_{i=1}^{m-1} \beta_i \log \gamma_i + \beta_m \log (1 - \sum_{i=1}^{m-1} \gamma_i)$$

Thus

$$\frac{\beta_i}{\gamma_i^0} - \frac{\beta_m}{\gamma_m^0} = 0 \quad i=1, \dots, m-1$$

or

$$\beta_i = \lambda \gamma_i \quad \text{for} \quad i=1, \dots, m.$$

But

$$\sum_{i=1}^m \beta_i = \sum_{i=1}^m \gamma_i^0 = 1,$$

thus

$$\gamma_1^0 = \beta_1 .$$

This shows the maximum of  $f$  occurs at the unique point  $c = (\beta_1, \dots, \beta_m)$ , as was to be proved.

Lemma 5.

Let  $\phi : R_+^m \rightarrow R$  be continuous, quasi-concave, 1-homogeneous and  $\phi(x) \geq 0$  for all  $x \in R_+^m$ . Then the following are equivalent

- (i)  $\phi$  is concave
- (ii)  $\phi$  is non-decreasing (i.e.  $\phi(x) \geq \phi(y)$  if  $x \geq y$ )
- (iii)  $\phi(x+y) \geq \phi(x) + \phi(y)$  for all  $x, y \in R_+^m$
- (iv)  $\phi(x) = 0$  for all  $x \in R_+^m$  or  $\phi(x) > 0$  for all  $x > 0$ .

Proof. We shall show that (i) implies (ii), (ii) implies (iii), (iii) implies (iv) and finally (iv) implies (i).

Suppose  $\phi$  is concave and for some  $x, y \in R_+^m$  we have  $x \leq y$  and  $\phi(x) > \phi(y)$ .

Let

$$z_k = ky + (1-k)x = k(y-x) + x \in R_+^m, \quad k=1, 2, \dots$$

Then

$$y = \frac{1}{k} z_k + (1 - \frac{1}{k}) x .$$

Thus by concavity

$$\phi(y) \geq \frac{1}{k} \phi(z_k) + (1 - \frac{1}{k}) \phi(x)$$

or

$$k [\phi(y) - \phi(x)] + \phi(x) \geq \phi(z_k) \geq 0 .$$

This is clearly a contradiction, since  $k [\phi(y) - \phi(x)]$  may be made arbitrarily close to  $-\infty$  by choosing  $k$  sufficiently large. Thus  $\phi$  is non-decreasing.

Secondly, suppose  $\phi$  is non-decreasing. If  $\phi(x)$  and  $\phi(y)$  are both positive then, using quasi-concavity of  $\phi$ , and since

$$\phi\left(\frac{x}{\phi(x)}\right) = 1 = \phi\left(\frac{y}{\phi(y)}\right) ,$$

we have

$$\begin{aligned} \frac{\phi(x+y)}{\phi(x)+\phi(y)} &= \phi\left(\frac{x+y}{\phi(x)+\phi(y)}\right) = \phi\left(\frac{\phi(x)}{\phi(x)+\phi(y)} \frac{x}{\phi(x)} + \frac{\phi(y)}{\phi(x)+\phi(y)} \frac{y}{\phi(y)}\right) \\ &\geq 1 . \end{aligned}$$

Thus  $\phi(x + y) \geq \phi(x) + \phi(y)$ .

If, then,  $\phi(x + y) < \phi(x) + \phi(y)$  it must be that either  $\phi(x) = 0$  or  $\phi(y) = 0$  but this would contradict the fact that  $\phi$  is non-decreasing. We have thus shown that (ii) implies (iii). To prove that (iii) implies (iv), suppose there is an  $x > 0$  with  $\phi(x) = 0$ . Now for any  $y \geq 0$  there is a  $\mu > 1$  such that

$$z = \mu x + (1-\mu)y \in R_+^m$$

(if  $y \leq x$  then  $z \in R_+^m$  for all  $\mu \geq 0$ , otherwise let  $y = (\eta_1, \dots, \eta_m)$ ,  $x = (\xi_1, \dots, \xi_m)$  and let  $\mu = \min_{\eta_1 > \xi_1} \frac{\eta_1}{\eta_1 - \xi_1} > 1$ ).

Now

$$x = \frac{1}{\mu} z + (1 - \frac{1}{\mu}) y$$

and thus

$$0 = \phi(x) \geq \phi(\frac{z}{\mu}) + \phi((\frac{\mu-1}{\mu})y)$$

hence  $\phi(\frac{\mu-1}{\mu} y) = \frac{\mu-1}{\mu} \phi(y) = 0$ , and  $\phi(y) = 0$ , thus showing that  $\phi(y) = 0$  for all  $y \in R_+^m$ . Finally, we show that (iv) implies (i).

If  $\phi(x) = 0$  for all  $x \in R_+^m$  then  $\phi$  certainly is concave. Assuming  $\phi(x) > 0$  for all  $x > 0$  and  $\phi$  is not con-

cave we have

$$\phi(\lambda x + (1 - \lambda)y) < \lambda\phi(x) + (1 - \lambda)\phi(y)$$

for some  $x, y \in R_+^m$  and  $\lambda \in (0, 1)$ . Let  $z = \frac{1 - \lambda}{\lambda} y$  then  $\phi(x + z) < \phi(x) + \phi(z)$ . As above, if  $\phi(x), \phi(y) > 0$ , we would have (using quasi-concavity of  $\phi$ )  $\phi(x + z) \geq \phi(x) + \phi(z)$ . Thus, say,  $\phi(z) = 0$  and we have

$$\phi(x + z) < \phi(x),$$

Taking a sequence  $z_k$  converging to  $z$  with each  $z_k > 0$ , we have  $\phi(x + z_k) \geq \phi(x) + \phi(z_k)$ , but by continuity of  $\phi$  we must also have  $\phi(x + z) \geq \phi(x) + \phi(z)$ . Completing the proof.

Lemma 6.

The C.P.U.  $\Psi$  is concave and continuous.

Proof. By Lemma 5 we see that each  $u_1$  is non-decreasing so that  $\Psi$  is non-decreasing,  $\Psi$  is also continuous since each  $u_1$  is, and of course  $\Psi$  is 1-homogeneous. It remains to show that  $\Psi$  is quasi-concave. Consider the functions  $f(\xi_1, \dots, \xi_m) = f(x) = \prod_1^{\beta_1} \xi_1$  defined for all  $x \geq 0$  and  $g(x) = \log f(x)$  defined for all  $x > 0$ . Now if  $g_{1k}$  denotes the 1, k partial derivative of  $g$  then

$$g_{1k}(x) = 0 \text{ if } 1 \neq k \text{ and } g_{11}(x) = -\frac{\beta_1}{\xi_1}.$$

Hence (see [5] p. 87, No. 35)  $g$  is concave and  $f$  is quasi-concave in the interior of  $R_+^m$ , but being continuous it must be quasi-concave over all of  $R_+^m$ . Now if

$$\psi(x_1, \dots, x_m) \leq \psi(y_1, \dots, y_m)$$

then

$$\begin{aligned} \psi(\lambda x_1 + (1-\lambda)y_1, \dots, \lambda x_m + (1-\lambda)y_m) &= \frac{m}{\pi} \left[ u_1(\lambda x_1 + (1-\lambda)y_1) \right]^{\beta_1} \\ &\geq \frac{m}{\pi} \left[ \lambda u_1(x_1) + (1-\lambda)u_1(y_1) \right]^{\beta_1} \geq \psi(x_1, \dots, x_m), \end{aligned}$$

completing the proof.

Lemma 7.

The C.U. function  $u$  is concave, continuous, 1-homogeneous and  $u(x) \geq 0$  all  $x \in R_+^n$ .

Proof. That  $u(x) \geq 0$  for all  $x \in R_+^n$  is trivial. To show homogeneity, let  $\lambda > 0$  and let

$$u(x) = \psi(x_1, \dots, x_m), \quad u(\lambda x) = \psi(y_1, \dots, y_m)$$

then

$$\sum_{i=1}^m x_i \leq x, \quad \sum_{i=1}^m y_i \leq \lambda x.$$

Thus

$$\sum_{i=1}^m \lambda x_i \leq \lambda x \quad \text{and} \quad \sum_{i=1}^m \frac{y_i}{\lambda} \leq x ,$$

whence it follows that

$$\Psi (\lambda x_1, \dots, \lambda x_m) \leq \Psi (y_1, \dots, y_m)$$

and

$$\Psi \left( \frac{y_1}{\lambda}, \dots, \frac{y_m}{\lambda} \right) \leq \Psi (x_1, \dots, x_m) .$$

But  $\Psi$  is 1-homogeneous so

$$\lambda \Psi (x_1, \dots, x_m) \leq \Psi (y_1, \dots, y_m) \leq \lambda \Psi (x_1, \dots, x_m)$$

or

$$\lambda u(x) \leq u(\lambda x) \leq \lambda u(x)$$

and

$$u(\lambda x) = \lambda u(x) .$$

We show next that  $u$  is super-additive (i.e. for any  $x, y \in \mathbb{R}_+^n$   
 $u(x + y) \geq u(x) + u(y)$ ).

Let

$$u(x) = \Psi (x_1, \dots, x_m)$$

$$u(y) = \Psi (y_1, \dots, y_m) .$$

Hence

$$\sum_{i=1}^m x_i \leq x , \quad \sum_{i=1}^m y_i \leq y ,$$

thus

$$\sum_{i=1}^m (x_i + y_i) \leq x + y$$

whence

$$u(x + y) \geq \Psi (x_1 + y_1, \dots, x_m + y_m) ,$$

but by Lemma 6  $\Psi$  is super-additive thus

$$u(x + y) \geq \Psi (x_1, \dots, x_m) + \Psi (y_1, \dots, y_m) = u(x) + u(y).$$

It is clear that homogeneity and super-additivity of  $u$  suffice to show that  $u$  is concave; it is of interest, however, and not difficult to demonstrate, that  $u$  is also continuous. Let  $x^k$  be a sequence in  $R_+^n$  converging to  $x$  in  $R_+^n$ . Let

$u(x^k) = \Psi(x_1^k, \dots, x_m^k)$  and let  $u(x) = \Psi(x_1, \dots, x_m)$ . Since the  $x_1^k$  are bounded we may assume, taking a subsequence if necessary, that the  $x_1^k$  converge to  $\bar{x}_1$  in  $R_+^n$ . Hence, by continuity of  $\Psi$ , the  $\Psi(x_1^k, \dots, x_m^k)$  converge to  $\Psi(\bar{x}_1, \dots, \bar{x}_m)$ . But  $\sum_{i=1}^m x_1^k \leq x^k$  hence  $\sum_{i=1}^m \bar{x}_1 \leq x$  and thus  $\Psi(\bar{x}_1, \dots, \bar{x}_m) \leq \Psi(x_1, \dots, x_m)$ . We thus see that  $\lim u(x^k) \leq u(x)$ .

Let  $x = (\xi_1, \dots, \xi_n)$ ,  $x^k = (\xi_1^k, \dots, \xi_n^k)$ ; we may assume that  $\xi_j^k > 0$  for all  $k$  whenever  $\xi_j > 0$ . Let

$$\lambda_k = \min_{\xi_j > 0} \left( \frac{\xi_j^k}{\xi_j} \right) \text{ (if } x = 0 \text{ let } \lambda_k = 1, \text{ all } k),$$

then  $\lambda_k > 0$  and  $\lambda_k$  converge to 1. But  $\lambda_k x \leq x^k$  so that  $\sum_{i=1}^m \lambda_k x_i \leq x^k$  (because we may assume that  $\sum_{i=1}^m x_i = x$ ). Thus

$$\Psi(\lambda_k x_1, \dots, \lambda_k x_m) = \lambda_k \Psi(x_1, \dots, x_m) \leq \Psi(x_1^k, \dots, x_m^k)$$

and, since the  $\lambda_k$  converge to 1, we have

$$\Psi(x_1, \dots, x_m) = u(x) \leq \lim u(x^k).$$

Hence  $u(x) = \lim_k u(x^k)$ , completing the proof.

We now proceed to prove Theorems 1 - 4.

Proof of Theorem 2. Suppose  $x_1 \in D_1(p)$  and  $\sum_{i=1}^m x_i = x$ .

By Lemma 3 we know there exist  $\alpha_1, \dots, \alpha_m$ , all non-negative, such that for each  $i=1, \dots, m$  we have

$$(5) \quad \alpha_i \beta_i = \alpha_i x_i p = u_i(x_i)$$

$$(6) \quad x_i p \leq \beta_i$$

$$(7) \quad \alpha_i y_i p \geq u_i(y) \quad , \quad \text{for every } y \in R_+^n$$

Note that  $\alpha_i > 0$  for each  $i=1, \dots, m$ , because otherwise we would have  $\phi_i(x) = 0$  for all  $x$ ; hence  $\beta_i = x_i p$ . Now

$$u(x) \geq \psi(x_1, \dots, x_m) \quad ,$$

if

$$u(x) = \psi(y_1, \dots, y_m) > \psi(x_1, \dots, x_m)$$

then

$$\begin{aligned} \sum_{i=1}^m (\alpha_i y_i p)^{\beta_i} &\geq \sum_{i=1}^m [u_i(y_i)]^{\beta_i} > \sum_{i=1}^m [u_i(x_i)]^{\beta_i} = \sum_{i=1}^m (\alpha_i x_i p)^{\beta_i} \\ &= \sum_{i=1}^m (\alpha_i \beta_i)^{\beta_i} \end{aligned}$$

thus

$$\sum_{i=1}^m (y_i p)^{\beta_i} > \sum_{i=1}^m \beta_i^{\beta_i} \quad .$$

But

$$\sum_{i=1}^m y_i \leq x ,$$

hence

$$\sum_{i=1}^m y_i p \leq px = 1$$

which contradicts Lemma 4. Thus  $u(x) = \Psi(x_1, \dots, x_m)$ .

Now if  $x \in D^*(p)$  and  $u(x) = \Psi(x_1, \dots, x_m)$ , let  $x = \sum_{i=1}^m x_i^0$  where  $x_i^0 \in D_1(p)$ . Using (5) - (7) and the fact that  $\Psi(x_1, \dots, x_m) = \Psi(x_1^0, \dots, x_m^0)$  (see first part of this proof) we have

$$\prod_{i=1}^m (\alpha_i \beta_i)^{\beta_i} = \prod_{i=1}^m [u_i(x_i^0)]^{\beta_i} = \prod_{i=1}^m [u_i(x_i)]^{\beta_i} \leq \prod_{i=1}^m [\alpha_i x_i p]^{\beta_i} .$$

Hence

$$\prod_{i=1}^m \beta_i^{\beta_i} \leq \prod_{i=1}^m (x_i p)^{\beta_i} ,$$

but  $\sum_{i=1}^m x_i \leq x$  so that  $\sum_{i=1}^m x_i p \leq xp = 1$ , and thus by Lemma 4

$$x_1 p = \beta_1 \quad \text{and} \quad u_1(x_1) = \alpha_1 \beta_1 .$$

Hence if  $y \in R_+^n$ , and  $yp \leq \beta_1$  then

$$u_1(y) \leq \alpha_1 yp \leq \alpha_1 \beta_1 = u_1(x_1) .$$

Thus  $x_1 \in D_1(p)$  for each  $i=1, \dots, m$ . As was to be shown.

Proof of Theorem 3. By Lemma 1 there exists  $q \in R_+^n$  such that

$$q \sum_{i=1}^m x_i = qx$$

and

$$\prod_{i=1}^m [u_i(x_i)]^{\beta_i} \geq \prod_{i=1}^m [u_i(y_i)]^{\beta_i} + q \sum_{i=1}^m (x_i - y_i)$$

for every  $y_1, \dots, y_m$  such that  $y_i \in R_+^n$ .

Let  $y_i = \lambda x_i$ , where  $\lambda \geq 0$  then

$$(1-\lambda) \prod_{i=1}^m [u_i(x_i)]^{\beta_i} \geq (1-\lambda) q \sum_{i=1}^m x_i = (1-\lambda) qx$$

thus

$$qx = \prod_{i=1}^m [u_i(x_i)]^{\beta_i} > 0.$$

Let  $p = \frac{q}{qx}$  then

$$(8) \quad 1 \geq \prod_{i=1}^m \left[ \frac{u_i(y_i)}{u_i(x_i)} \right]^{\beta_i} + p \sum_{i=1}^m (x_i - y_i) \quad \text{all } y_i \in R_+^n.$$

For a fixed  $k$ , let  $y_i = x_i$  if  $i \neq k$ ,  $y_k = \lambda x_k$  where  $\lambda \geq 0$ .

Thus  $1 \geq \lambda^{\beta_k} + p x_k (1-\lambda)$  hence by Lemma 2

$$\beta_k = p x_k \text{ for every } k=1, \dots, m.$$

Now for a given  $k$  let, in (8),  $y_i = x_i$  if  $i \neq k$  and  $y_k = x$  then

$$(9) \quad 1 \geq \left[ \frac{u_k(x)}{u_k(x_k)} \right]^{\beta_1} + p(x_k - x) \quad \text{for all } x \in R_+^n$$

Thus, if  $x \in R_+^n$  and  $x p \leq \beta_k = x_k p$  then

$$1 \geq \left[ \frac{u_k(x)}{u_k(x_k)} \right]^k, \quad \text{or} \quad u_k(x) \leq u_k(x_k).$$

Thus  $x_k \in D_k(p)$  for  $k=1, \dots, m$ .

Proof of Theorem 1. Suppose  $x \in D^*(p)$ ; we wish to show that for any  $y \in R_+^n$ , if  $yp \leq 1$  then  $u(y) \leq u(x)$ . Let  $x = \sum_{i=1}^m x_i$  where  $x_i \in D_i(p)$ , and let  $u(y) = \Psi(y_1, \dots, y_m)$  hence  $\sum_{i=1}^m y_i \leq y$  and thus  $\sum_{i=1}^m y_i p \leq yp \leq 1$ . Thus by Lemma 4, and using (5) - (7) we have

$$\begin{aligned} u(x) &= \Psi(x_1, \dots, x_m) = \prod_{i=1}^m [u_i(x_i)]^{\beta_1} = \prod_{i=1}^m [\alpha_i \beta_i]^{\beta_1} \geq \prod_{i=1}^m (\alpha_i y_i p)^{\beta_1} \\ &\geq \prod_{i=1}^m [u_i(y_i)]^{\beta_1} = \Psi(y_1, \dots, y_m) = u(y). \end{aligned}$$

Hence  $u(x) \geq u(y)$ .

Conversely, suppose  $x$  maximizes  $u$  subject to  $x \geq 0$ ,  $px \leq 1$ . Let  $u(x) = \psi(x_1, \dots, x_m)$  then  $\sum_{i=1}^m x_i \leq x$ . Also  $p \neq 0$ , for otherwise  $u(y) = 0$  for all  $y$  which is clearly not the case, thus we apply Lemma 3 which tells us that there is an  $\eta \geq 0$  such that

$$\eta px = \eta = u(x) > 0 \quad \text{and}$$

$$\eta py \geq u(y) \quad \text{for all } y \in R_+^n$$

Now for any  $y_1, \dots, y_m \in R_+^n$  we have:

$$(9) \quad \eta p \sum_{i=1}^m y_i \geq u\left(\sum_{i=1}^m y_i\right) \geq \psi(y_1, \dots, y_m) = \sum_{i=1}^m [u_i(y_i)]^{\beta_i}$$

thus

$$(10) \quad p \sum_{i=1}^m y_i \geq \sum_{i=1}^m \frac{[u_i(y_i)]^{\beta_i}}{[u_i(x_i)]^{\beta_i}}, \quad \text{for all } y_1, \dots, y_m \in R_+^n$$

Hence for every  $i=1, \dots, m$  and every  $\lambda \geq 0$  we have

$$1 - \lambda^{\beta_i} \geq (1 - \lambda)p x_i$$

thus, by Lemma 2,  $\beta_i = p x_i$ .

Now if  $y \in R_+^n$  and  $yp \leq \beta_k$  then from (10) we get

$$\left[ \frac{u_k(y)}{u_k(x_k)} \right]^{\beta_k} < p \left[ \sum_{i \neq k} x_i + y \right] \leq 1 - x_k p + yp \leq 1$$

or  $u_k(y) \leq u_k(x_k)$ . Thus  $x_i \in D_1(p)$  for all  $i=1, \dots, m$ .

But  $px = 1 \geq \sum_{i=1}^m px_i = 1$ , thus  $px = \sum_{i=1}^m px_i$  hence  $x \in D^*(p)$ .

Proof of Theorem 4. Let  $p, q$  be equilibrium price vectors for the bundle  $x$ . Let  $x_1, \dots, x_m$  and  $y_1, \dots, y_m$  be their corresponding equilibrium distributions. Let  $\alpha_1, \bar{\alpha}_1$  be the corresponding numbers appearing in (5) - (7). Then  $\alpha_1, \bar{\alpha}_1 > 0$  for every  $i$  and

$$\left( \sum_1 \frac{\alpha_1}{\bar{\alpha}_1} \beta_1 \right) \left( \sum_1 \frac{\bar{\alpha}_1}{\alpha_1} \beta_1 \right) = \sum_1 \beta_1^2 + \sum_{i < k} \left( \frac{\alpha_1}{\bar{\alpha}_1} + \frac{\bar{\alpha}_k}{\alpha_k} \right) \beta_1 \beta_k \leq$$

$$\leq \sum_1 \beta_1^2 + 2 \sum_{i < k} \beta_1 \beta_k = \left( \sum_1 \beta_1 \right)^2 = 1.$$

Thus, say,  $\sum_1 \frac{\alpha_1}{\bar{\alpha}_1} \beta_1 \leq 1$ .

By Lemma 4 we then have

$$\prod_{i=1}^m \left[ \frac{\alpha_1}{\bar{\alpha}_1} \beta_1 \right]^{\beta_1} = \left[ \prod_{i=1}^m \left( \frac{\alpha_1}{\bar{\alpha}_1} \right)^{\beta_1} \right] \prod_{i=1}^m \beta_1^{\beta_1} < \prod_{i=1}^m \beta_1^{\beta_1}$$

hence  $\prod_{i=1}^m \alpha_1^{\beta_1} < \prod_{i=1}^m \bar{\alpha}_1^{\beta_1}$ . But

$$\alpha_1 \beta_1 = u_1(x_1), \quad \bar{\alpha}_1 \beta_1 = u_1(y_1)$$

and by Theorem 2\*

$$\prod_{i=1}^m \left[ u_1(x_1) \right]^{\beta_1} = \prod_{i=1}^m \left[ u_1(y_1) \right]^{\beta_1}$$

hence

$$\sum_{i=1}^m \alpha_i \beta_i = \sum_{i=1}^m \bar{\alpha}_i \beta_i$$

and by Lemma 4

$$\frac{\alpha_1}{\bar{\alpha}_1} \beta_1 = \beta_1, \quad \text{or} \quad \alpha_1 = \bar{\alpha}_1.$$

Consequently  $u_1(x_1) = u_1(y_1)$  . Q.E.D.

Conclusion. We have shown that under the assumptions listed in the first paragraph of Section 2:

- (i) A concave, homogeneous and continuous community-utility function exists.
- (ii) A social welfare function is given by  $\prod_{i=1}^m [u_i(x_i)]^{\beta_i}$  or equivalently by  $\sum_{i=1}^m \beta_i \log u_i(x_i)$ . This function is concave hence:
- (iii) Equilibrium distributions and prices may be characterized as the primal and dual variables, respectively, of a concave programming problem with linear constraints, the constraints being the usual market budget inequalities (i.e., the requirement that none of the goods be over-demanded). The objective function is, of course, the social welfare function given in (ii).
- (iv) While equilibrium prices and distributions need not be unique, the pay-offs are.

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